MAST602

Lecture 4

Equations of continuity & motion

Continuity, diffusion, box models
Estuarine convective flow
Equations of motion
Pressure gradient force
Coriolis force
Gravity
Friction
Shearing stress
Eddy viscosities

MAST 602

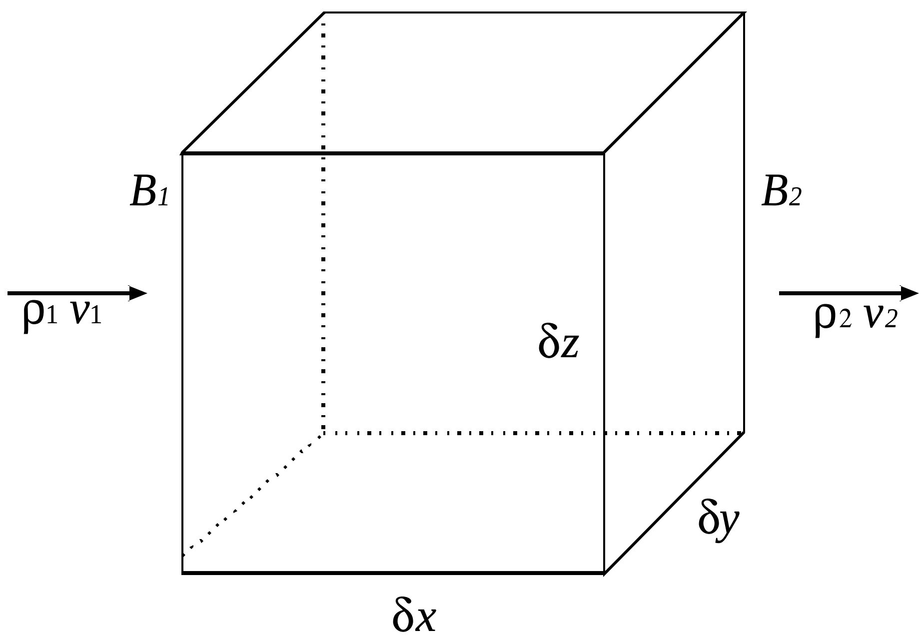
Lecture 4
Equations of continuity and motion

Continuity

Ref.: Knauss, Chapter 4

Recall that, for heat,
heat in = heat out

Or, more generally,
change = what goes in – what goes out



We let flow in be negative,
flow out be positive.

Look at a cubic volume of fluid,
of volume V, with
the area of face $i = B_i$
the density of fluid across face $i = r_i$
the velocity across face $i = v_i$

The rate of change of mass (density) is

$$V \frac{\partial \rho}{\partial t} = -\sum_{i=1}^{6} B_i \, \rho_i \, v_i$$
$$= 0$$

for steady state, if there is no source or sink.

Similarly, for salt, we can write:

$$V \frac{\partial S}{\partial t} = -\sum_{i=1}^{6} B_i \, S_i \, v_i$$
$$= 0$$

for steady state.

i.e., for $\frac{\partial S}{\partial t} = 0$

ρ and S are *conservative* properties:
i.e., there are no sources
or sinks in the fluid.

Non-conservative properties
can change values,
independent of the flux.

Examples are oxygen
phosphate
radioactive materials

Thus, for a non-conservative property

$$V \frac{\partial H}{\partial t} = \underbrace{-\sum_{i=1}^{6} B_i H_i v_i}_{\text{flux across boundaries}} \; - \underbrace{V \gamma H}_{\text{depletion}}$$

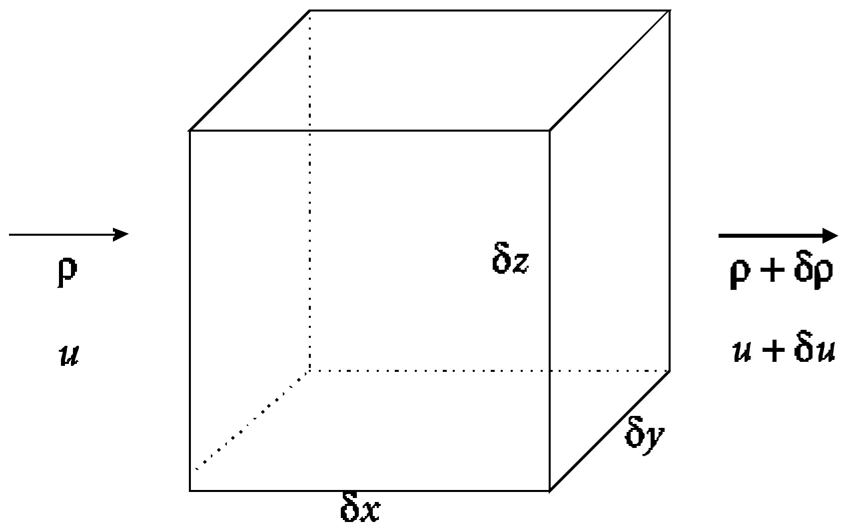
H = concentration of radioactive material
γ = rate of decay of the material

Or,

$$V \frac{\partial O}{\partial t} = -\sum_{i=1}^{6} B_i O_i \, v_i - \hat{O} V$$

O = oxygen concentration
$\hat{O}$ = rate of oxygen depletion or production

To develop the continuity equation,
consider a small volume of fluid,
of dimensions *dx, dy, dz*:



Mass flow *into* the volume is

$$\rho\, u\, dy\, dz$$

and mass flow *out* of the volume is

$$\left(\rho + \underbrace{\frac{\partial \rho}{\partial x} dx}_{d\rho}\right)\left(u + \underbrace{\frac{\partial u}{\partial x} dx}_{du}\right) dy\, dz$$

The net flow out of the volume
in the x-direction is the difference:

$$\left[u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} + \frac{\partial \rho}{\partial x}\frac{\partial u}{\partial x} dx\right] dx\ dy\ dz = \left[\frac{\partial(\rho u)}{\partial x} + O(dx)\right] dx\ dy\ dz$$

As we take $dx \Rightarrow 0$, the second term in
the brackets becomes negligible, and
for the *x*-direction, we have simply

$$\frac{\partial\ (\rho\ u)}{\partial\ x}\, dx\, dy\, dz \qquad \text{for the } x\text{-direction}$$

as the net flow out of the volume.

For flow in all three directions,
total flow out of the volume is

$$\left[\frac{\P(r\,u)}{\P\,x}+\frac{\P(r\,v)}{\P\,y}+\frac{\P(r\,w)}{\P\,z}\right]d\,x\,d\,y\,d\,z$$

v and w are the velocity components
in the y and z directions.

The change of mass in the volume
per unit time is

$$\frac{\P\,r}{\P\,t}\,d\,x\,d\,y\,d\,z$$

And if mass is to be conserved,
the change of mass in the volume
must balance the total flow out:

$$\underbrace{\frac{\P\,r}{\P\,t}}_{\text{change}}=\underbrace{-\frac{\P}{\P\,x}(r\,u)-\frac{\P}{\P\,y}(r\,v)-\frac{\P}{\P\,z}(r\,w)}_{\text{flow out}}$$

This is the equation of continuity of volume.

[change = what comes in - what goes out]

We can usually assume that the ocean is
incompressible. That is, the density,
r, is constant, and thus

$$\frac{\P\,r}{\P\,t}=\frac{\P\,r}{\P x}=\frac{\P\,r}{\P y}=\frac{\P\,r}{\P\,z}=0$$

Which gives the equation of continuity
for incompressible flow:

$$\boxed{\frac{\P\,u}{\P\,x}+\frac{\P\,v}{\P\,y}+\frac{\P\,w}{\P\,z}=\;0}$$

Salt

A similar continuity equation can be derived for salt.

Let the mass of salt per unit volume be ρS.

As above,

$$\frac{\partial S}{\partial t} = -\frac{\partial (Su)}{\partial x} - \frac{\partial (Sv)}{\partial y} - \frac{\partial (Sw)}{\partial z}$$

This becomes:

$$\frac{\partial S}{\partial t} = -u\frac{\partial S}{\partial x} - v\frac{\partial S}{\partial y} - w\frac{\partial S}{\partial z}$$

Non-conservative properties

Similarly, for oxygen, or other non-conservative properties, we can write

$$\frac{\partial H}{\partial t} = -u\frac{\partial H}{\partial x} - v\frac{\partial H}{\partial y} - w\frac{\partial H}{\partial z} - \gamma H$$

Radioactive decay

$$\frac{\partial O}{\partial t} = -u\frac{\partial O}{\partial x} - v\frac{\partial O}{\partial y} - w\frac{\partial O}{\partial z} - \hat{O}$$

oxygen depletion

Diffusion

In addition to advection, properties can change on a variety of other length scales, from molecular to large scale.

Diffusion can occur through molecular processes:

even though the fluid is motionless, transfer of properties can take place.

Diffusion can occur by mixing

Heat, salt, and momentum fluxes are proportional to the gradient of the property.

For salt:

$$F_s = - r\, k_s \frac{\P\, S}{\P\, n} \qquad \text{kg m}^{-2}\ \text{s}^{-1}$$

where $k_S \cong 1.5 \times 10^{-9}\ \text{m}^2\text{s}^{-1}$ coefficient of molecular diffusivity of salt

For heat: $Q = - r\, k_Q\, c_p \frac{\P\, T}{\P\, n}\ \text{J m}^{-2}\ \text{s}^{-1}$ [$\text{J m}^{-2}\ \text{s}^{-1} = \text{W m}^{-2}$]

where $k_Q \cong 1.5 \times 10^{-7}\ \text{m}^2\ \text{s}^{-1}$ coefficient of molecular diffusivity of heat

and $c_p \cong 4000$ J/Kg C

specific heat at constant pressure

For momentum:

$$t = - m \frac{dv}{dn} \qquad \text{kg m s}^{-1}\ \text{m}^{-2}\text{s}^{-1}$$

where $m \cong 1 \times 10^{-3}\ \text{kg m}^{-1}\ \text{s}^{-1}$ molecular viscosity

[t is also called the frictional stress]

In each case n is the direction down
the gradient of the property,
normal to the isopleths

The molecular coefficients are
properties of the *fluid*

Note that heat diffuses more rapidly than salt

Molecular diffusion is
orders of magnitude too small to
account for the observed diffusion.

Instead, let's look at turbulent diffusion,
or mixing:

Turbulent diffusion

In analogy, the turbulent eddy coefficients
(or eddy diffusivities) might be:

$A_z \sim 10^{-4} \Rightarrow 10^{-3}\ \text{m}^2\ \text{s}^{-1}$
vertical eddy diffusivity

$A_h \sim 10^{-1} \Rightarrow 10^{3}\ \text{m}^2\ \text{s}^{-1}$
horizontal eddy diffusivity

So that the turbulent diffusion terms are

$$-A_z \frac{\partial S}{\partial z}, \quad -A_h \frac{\partial S}{\partial x}, \quad -A_h \frac{\partial S}{\partial y}$$

Note that:

- These coefficients are properties of the *flow*
- $A_h >> A_z$, because of stable stratification
- $A_z \sim 10^4 \times \mu$
  i.e., eddy diffusion is 10,000 times
  more effective than molecular diffusion!

We can now add the turbulent terms
to the equation of continuity.

The balance of fluxes is:

$$V \frac{\partial S}{\partial t} = -\underbrace{\sum_{i=1}^{6} B_i S_i v_i}_{\text{advection}} + \underbrace{\sum_{i=1}^{6} A_i B_i \left(\frac{\partial S}{\partial n}\right)_i}_{\text{turbulence}}$$

$$V \frac{\partial O}{\partial t} = -\underbrace{\sum_{i=1}^{6} B_i O_i v_i}_{\text{advection}} + \underbrace{\sum_{i=1}^{6} A_i B_i \left(\frac{\partial O}{\partial n}\right)_i}_{\text{turbulence}} \underbrace{-\hat{O} V}_{\text{depletion}}$$

Then the continuity equations
for salt and oxygen become:
[Knauss eqns. 4.17, 4.18]

$$\frac{\partial S}{\partial t} = -\frac{\partial}{\partial x}(Su) - \frac{\partial}{\partial y}(Sv) - \frac{\partial}{\partial z}(Sw) + \underbrace{A_h \left(\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2}\right)}_{\text{horizontal turbulent flux}} + \underbrace{A_z \left(\frac{\partial^2 S}{\partial z^2}\right)}_{\text{vertical flux}}$$

$$\frac{\partial O}{\partial t} = -\frac{\partial}{\partial x}(Ou) - \frac{\partial}{\partial y}(Ov) - \frac{\partial}{\partial z}(Ow) + \underbrace{A_h \left(\frac{\partial^2 O}{\partial x^2} + \frac{\partial^2 O}{\partial y^2}\right)}_{\text{horizontal turbulent flux}} + \underbrace{A_z \left(\frac{\partial^2 O}{\partial z^2}\right)}_{\text{vertical flux}} - \underbrace{\hat{O}}_{\text{depletion}}$$

$\boxed{\text{Ref: Knauss, Box 4.2, p. 74.}}$

[Note: For Knauss eqn. 4.14 (p. 73), the value given should be 10^{-4}, not 10^{4}!]

Box Models

Let's apply the continuity equation
to a practical oceanographic problem
by dividing the ocean into a series of
boxes or layers.

We consider the flux of material into and
out of these boxes.

Consider a box of volume *V*,
filled with water of
constant density and salinity.

Water flows into and out of the box
by *advection* and/or *diffusion*

Then, for mass and salt,

$$-\sum_{i=1}^{6} B_i \, r_i \, v_i + \sum_{i=1}^{6} A_i \, B_i \left(\frac{\P \, r}{\P \, n} \right)_i = 0$$

$$-\sum_{i=1}^{6} B_i \, S_i \, v_i + \sum_{i=1}^{6} A_i \, B_i \left(\frac{\P \, S}{\P \, n} \right)_i = 0$$

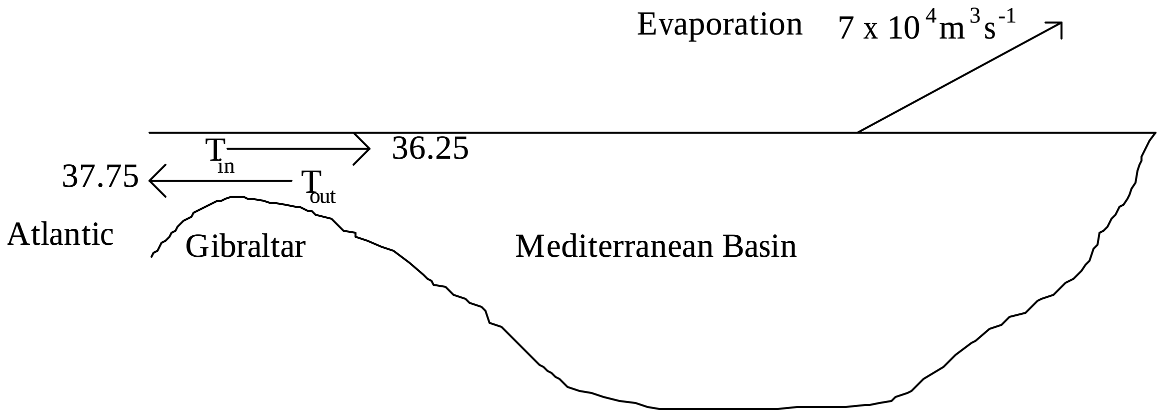
where Brv = mass per unit time
= mass transport

and Bv = volume per unit time
= volume transport, T

so, for steady state,

$$\sum T_i = 0$$
$$\sum T_i S_i = 0$$

We can apply these equations
to the Mediterranean Sea.



By measurement, we know that
the salinity of *outflowing* (Mediterranean) water is 37.75 psu

Salinity of *incoming* (Atlantic Ocean) water is 36.25 psu

Estimates give

Evaporation $\sim 7 \times 10^4$ m^3 s^{-1}

so, $T_{in} = T_{out} + 70{,}000$ m^3 s^{-1}

and $T_{in} S_{in} - T_{out} S_{out} = 0$

Solving these equations, we get

$T_{in} = 1.75 \times 10^6$ m^3 s^{-1}

$T_{out} = 1.68 \times 10^6$ m^3 s^{-1}

How long will a particle of water "stay" in the Mediterranean?

We can define a "residence time", Ξ, as the average time a molecule of water will spend in the basin, even though many parts of the Mediterranean Sea are likely to be undisturbed in spite of the exchanges through the Straits of Gibraltar and the processes of evaporation and precipitation.

$$\Xi = \frac{V}{T_{in}}$$

The volume, *V*, of the Mediterranean Sea
is 4×10^{15} m^3

So the calculation gives

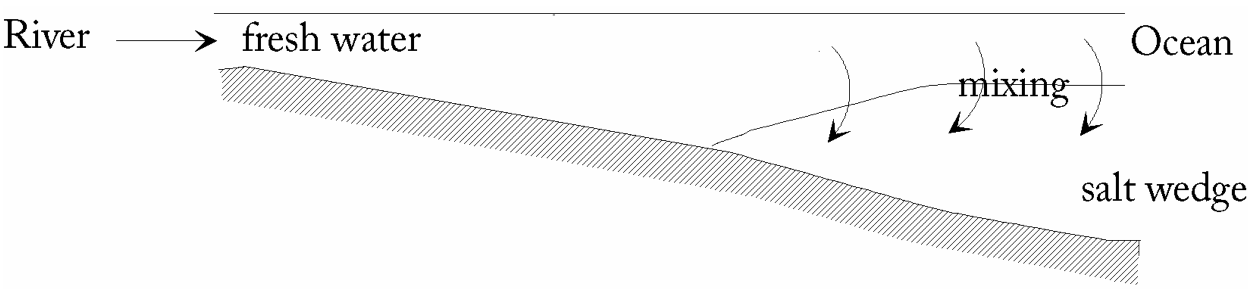
$$\Xi = \frac{4\times10^{15}\mathrm{m}^3}{1.75\times10^6\mathrm{m}^3\mathrm{s}^{-1}}$$
$$= 2.2\times10^9\mathrm{s}$$
$$= 70 \text{ years}$$

This simple calculation is of
questionable physical significance.

Estuarine convective flow

Ref: Knauss, Chapter 11

The procedure just applied to the Mediterranean Sea is similar to how some estuarine convective systems can be treated. []



If there were no mixing,
estuarine flow would be like a river.

However, there *is* mixing...

- saltwater is mixed upward
- freshwater is mixed downward

This mixing:

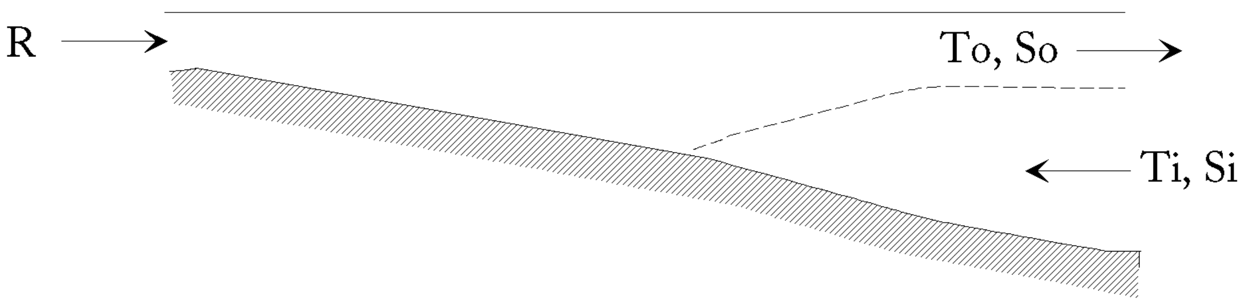
- reduces salinity $\left.\right\}$ at inner edge of the
- reduces density $\quad$ salt-water wedge
- establishes a horizontal pressure gradient
- thus drives the wedge further up the estuary
- this increases the shear and
  thus increases the mixing
- etc., the process continues…

The net result of this process is

- inflow at the bottom [salt wedge]
- outflow at the surface [river water]

Let's do a simple conservation model:

Volume transports in and out are T_i and T_0
The volume of river inflow is R
The salinities in and are S_i and S_0



For the conservation of volume:

$$T_o = T_i + R$$

For the conservation of salt:

$$T_o S_o = T_i S_i$$

solving, we get:

$$T_i = \frac{S_o}{S_i - S_o} R$$

$$T_o = \frac{S_i}{S_i - S_o} R$$

Note that,

If S_i - S_o decreases, T_i and T_o increase

If R increases, T_i and T_o increase.

So, in this simple model,
increasing the river flow
$\Rightarrow$ increases the bottom inflow!

The actual driving mechanism is pressure:
downstream at the surface, upstream below
due to the slope of the interface
and/or density decrease up the estuary.

Estuarine residence time

We may also define a "residence time" for water in an estuary.

The flushing time of estuaries can be an estimate of value in studying pollution. It can be estimated from the fresh-water inflow and the fresh-water volume in the estuary.

An estimate of the fresh water volume, V_{fw}, is

$$V_{fw} = V_e \frac{(S_o - S_e)}{S_o}$$

where

V_e is the volume of the estuary
S_o is the salinity at the mouth of the estuary
S_e is the weighted-mean salinity in the estuary

An estimate of the flushing time is then

$$X = \frac{V_e}{R}\frac{(S_o - S_e)}{S_o}$$

as before, R is the river input

This is easy to write down in class notes, but may be difficult to apply:

A problem in estimating flushing time with this method is obtaining enough data to estimate the volume-weighted mean salinity in the estuary.

Equations of motion

Ref.: Knauss, Chapter 5

Mathematical relationships are used
to represent motion:
they are approximations of
currents, waves, tides, turbulence…

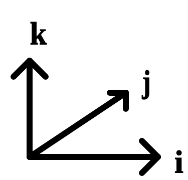
The coordinate system uses
right-handed cartesian coordinates:

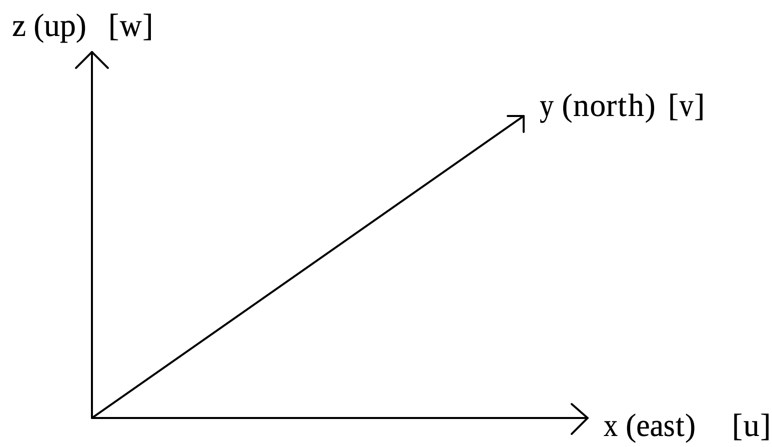
- z is up
- x is east
- y is north
- unit vectors are **i, j, k**

Watch out for two possible sources of confusion:

1. In meteorology, a *northerly wind*
   blows *from* the north
   In oceanography a *north current*
   flows *to* the north

2. Is z up or is z down?
   Look at a couple of classic texts:
   - it's up in (Pond and Pickard 1983)
   - it's down in (Sverdrup, Johnson et al. 1942)
   - it's mostly up in Knauss (but not always!)
   We'll take it up in mast602





Newton's second law

Newton's second law of motion:

$F = m\,a$

A resultant force **F**,
acting on a body of mass m,
will produce an acceleration
(a rate of change of velocity), **a**.

Boldface characters indicate vectors.

If it is the result of several forces,
we take the vector sum.

If **F** = 0,
a = 0,
and there is no change in velocity.

So,

$\mathbf{a} = \mathbf{F}/m$

or,

$$\frac{d\,\mathbf{u}}{d\,t} = \frac{1}{m}\sum \mathbf{F}$$

For fluids, where **F** is the
force per unit volume,

$$\frac{d\,\mathbf{u}}{d\,t} = \frac{1}{r}\sum \mathbf{F}$$

The vector equation can be re-written,
in cartesian coordinates:

$$\frac{du}{dt} = \frac{1}{r}\sum F_x$$

$$\frac{dv}{dt} = \frac{1}{r}\sum F_y$$

$$\frac{dw}{dt} = \frac{1}{r}\sum F_z$$

We will introduce each of the principal forces
in the equation of motion,
one by one.

These forces include:

- Pressure (gradient) force
- Coriolis force
- Gravity
- Friction

So that,
acceleration =
(pressure gradient + coriolis + gravity
+ friction + other) forces/unit mass

Note that,
particle acceleration
= local acceleration + field acceleration

u is a function of x, y, z, & t, so:

$$\underbrace{\frac{du}{dt}}_{\text{particle acceleration}} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$$

$$= \underbrace{\frac{\partial u}{\partial t}}_{\text{local acceleration at a point}} + \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}_{\text{field acceleration, moving with the fluid particles}}$$

Local acceleration vs. field acceleration:
Example: the waterfall:

$$\frac{\partial u}{\partial t} = 0$$

$$\frac{du}{dt} \neq 0$$

For further information, see Knauss, p. 83, Box 5.1

Pressure gradient force

See Knauss, p. 85, Box 5.2

A particle moves from high to low pressure. So:

$$\frac{du}{dt} = -\frac{1}{r}\frac{\P\, p}{\P\, x} + \cdots$$

$$\frac{dv}{dt} = -\frac{1}{r}\frac{\P\, p}{\P\, y} + \cdots$$

$$\frac{dw}{dt} = -\frac{1}{r}\frac{\P\, p}{\P\, z} + \cdots$$

or, in vector form,
the pressure gradient force is

$$-\frac{1}{r}\left(\frac{\P\, p}{\P\, x}\mathbf{i} + \frac{\P\, p}{\P\, y}\mathbf{j} + \frac{\P\, p}{\P\, z}\mathbf{k}\right) = -a\,\nabla p$$

Where

$$a = \frac{1}{r} \qquad \text{[specific volume]}$$

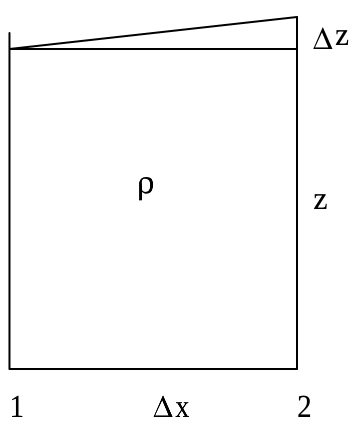
and the “gradient operator” is

$$\nabla \equiv \mathbf{i}\frac{\P}{\P\, x} + \mathbf{j}\frac{\P}{\P\, y} + \mathbf{k}\frac{\P}{\P\, z}$$

Example: sloping water surface

$p_1 = r\, g\, Z$

$p_2 = r\, g\,(Z + DZ)$

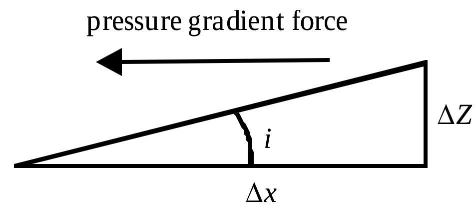


The pressure gradient is

$$\frac{1}{r}\frac{\P\ p}{\P\ x} = \frac{1}{r}\frac{p_2 - p_1}{\Delta x}$$

$$= g\frac{\Delta Z}{\Delta x}$$

$$= g\ i_x$$



$$i_x = \frac{\Delta z}{\Delta x}$$

where i_x is the slope in the x-direction

That is, a pressure gradient force can arise from a slope of the sea surface, *i*.

Note That the pressure gradient force is directed *down* the slope.

Coriolis force

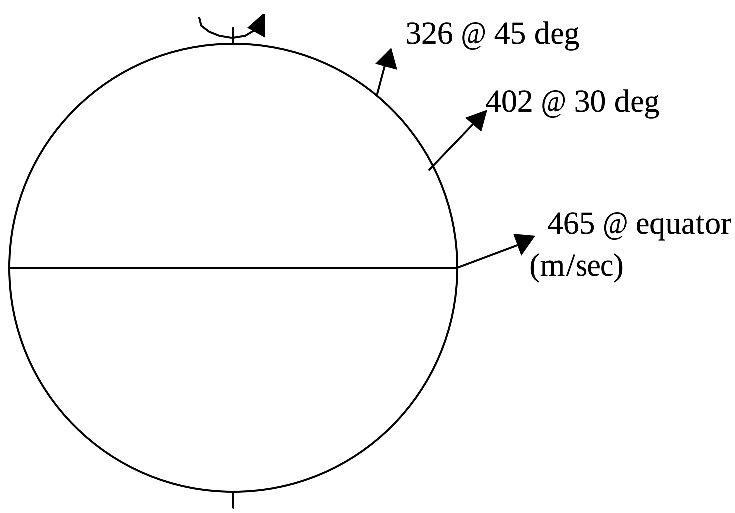
Coriolis is a non-intuitive force, difficult to comprehend.

See Knauss, p. 90. Box 5.3

Particles are accelerated on an earth that itself is rotating about its axis.

Let's look at two examples (maybe having some intuitive value)

Example 1. North-south movement of a particle of water on the surface:



Example 2. Pendulum at the north pole:

- the pendulum appears to rotate clockwise @ 15^{o} per hour
- in 12 hours it turns through 180^{o}
- the rotation is counterclockwise at the South Pole
- at the equator, there will be no rotation
- at any latitude, ϕ, one can show that
  the pendulum rotates 180^{o} in a time
  $T = \dfrac{12hr}{\sin f}$ [half pendulum day]

Such a pendulum is called
Foucault's Pendulum

(there's one in the Smithsonian Institution)

The equation of motion
relative to the axes of the earth is:

$$\left(\frac{d\,\mathbf{V}}{d\,t}\right)_e = \underbrace{-a\,\nabla\,p}_{\text{pressure}} - \underbrace{2\Omega\times\mathbf{V}}_{\text{coriolis}} + \underbrace{\mathbf{g}_f}_{\text{gravity}} - \underbrace{\Omega\times(\Omega\times\mathbf{R})}_{\text{centripetal}} + \underbrace{\mathbf{F}}_{\substack{\text{others, e.g.,}\\ \text{friction}}}$$

where

R is the vector distance to the center of the earth,
Ω is the angular rotation of Earth
$(\,)_f$ = relative to fixed coordinates
$(\,)_e$ = relative to the earth

See Knauss, Box 5.3, Page 90, for details

The term $2W \times \mathbf{V}$
is called the *coriolis acceleration*

The term - $W \times (W \times \mathbf{R})$
is sometimes called the *centrfugal acceleration*.
It acts opposite to gravity,

thus, the acceleration due to gravity,

$$\boldsymbol{g} = \boldsymbol{g}_f - W \times (W \times \mathbf{R})$$

In non-vector terms, the
    coriolis acceleration term, $2W \times$ **V is**

In the east (x) direction:

$$2\Omega v \sin f - 2\Omega w \cos f$$
$$= fv - 2\Omega w \cos f$$

In the north (y)direction:

$$-2\Omega u \sin f$$
$$= - fu$$

In the up (z) direction:

$$2\Omega u \cos f$$

where Ω is the angular velocity of the earth,
    = 2π/24hr
    = 7.29×10^{-5} s^{-1}

and $f = 2\ W \sin\ f$,   the *coriolis parameter*

f = latitude
    = -90° at the south pole
    = +90° at the north pole

- The coriolis force
    is proportional to velocity relative to the earth
    (no **v**, no coriolis force)
- The coriolis force increases with latitude
    it's maximum at the poles
    it's zero at the equator
- The coriolis force
    acts at a right angle to the motion:
    to the right in the northern hemisphere
    to the left in the southern hemisphere
- The term 2W w cos ϕ is small
    (usually $w \ll u$, v and is ignored)
- The term 2 W u cos ϕ
    is called the Eötvös correction to gravity.
    it's proportional to eastward velocity.
    it will affect gravity measurements
        taken on a moving ship.
- The coriolis parameter, f,
    is positive in the northern hemisphere,
    negative in the southern hemisphere.

Gravity

- acts along the z axis
- slight variations in gravity are usually ignored by physical oceanographers
- gravity increases with depth in the ocean:

  $g(z) = g_0 - 2.3 \times 10^{-6}\, z$ (g in m s^{-2}; z in m.)

  [the minus sign, since z is measured upward. Knauss (eqn. 5.20), doesn't use a minus sign]

- For the purposes of mast 602, take $g = 9.81$ m s^{-2}
- The gravity term enters only into the z-component of the equation of motion
- The term is negative, since it operates in the negative z-direction.

The equation of motion now can be written:

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega \sin\phi\, v - 2\Omega\cos\phi\, w + F_x$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega \sin\phi\, u + F_y$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + 2\Omega \cos\phi\, u - g + F_z$$

In vector notation,
the equation of motion is:

$$\frac{d\mathbf{V}}{dt} = \left(\frac{\partial \mathbf{V}}{\partial t}\right)_e + (\mathbf{V}\cdot\nabla)\mathbf{V} = -\alpha\,\nabla p - 2\Omega\times\mathbf{V} + \mathbf{g} - \Omega\times(\Omega\times\mathbf{R}) + \mathbf{F}$$

where **F** represents other forces,
principally friction.

Let's look at friction more closely.

Friction

Water viscosity converts kinetic energy to heat

If no new energy were added to the ocean,
the ocean would eventually “run down”

But the total kinetic energy is ~ 0.01°C

A simple way to incorporate friction is to
add a term proportional to velocity:

$$\left.\begin{array}{l} \text{friction } (x) = -Ju \\ \text{friction } (y) = -Jv \\ \text{friction } (z) = -Jw \end{array}\right\} \text{(probably } \textit{too} \text{ simple)}$$

This formulation says nothing about
the physical processes involved

There are two ways of describing the molecular viscosity

- dynamic viscosity, $\mu \cong 1 \times 10^{-3}$ kg m^{-1} s^{-1}
- kinematic viscosity, $\nu \cong 1 \times 10^{-6}$ m^2 s^{-1}

  where $n = \frac{m}{r}$

As already discussed, the rate of momentum transfer is

$$t = m\frac{\P u}{\P z}$$

Molecular viscosity can't account
for what's observed

Turbulent viscosity: transfers momentum

- is it analogous to molecular friction?
  - globs of fluid $\Rightarrow$ molecules
  - mixing lengths $\Rightarrow$ free path lengths
- the molecular analogy can't cope
  with the range of scales:
  from centimeters to many kilometers
- transfer of kinetic energy to heat
  - ~ $10^{-6} \rightarrow 10^{-5}$ W kg^{-1}

Shearing stress

See Knauss, p. 100, Box 5.4

Example: wind over a water surface.
momentum is transferred
vertically down into the fluid

$$t_{xz} = m \frac{\P\, u}{\P\, z}$$

where t_{xz} = shearing stress tensor

the first subscript denotes
the direction of the stress

the second subscript denotes
the plane in which the stress acts

Thus, we can write:

$$t_{xy} = m \frac{\P\, u}{\P\, y}$$

$$t_{xx} = m \frac{\P\, u}{\P\, x}$$

Force is stress $\times$ surface area.

In the x direction,
the elements of the stress tensor are

$$t_{xz} = m \frac{\P\, u}{\P\, z}$$

$$t_{xy} = m \frac{\P\, u}{\P\, y}$$

$$t_{xx} = m \frac{\P\, u}{\P\, x}$$

The stresses produce forces,
for example in the x-direction:

$$\frac{du}{dt} = \frac{1}{r} \frac{\P\, t_{xz}}{\P\, z}$$

$$= \frac{1}{r} \frac{\P}{\P\, z} \left(m \frac{\P\, u}{\P\, z} \right)$$

$$= \frac{m}{r} \frac{\P^2 u}{\P\, z^2}$$

With all the components,

$$\frac{du}{dt} = \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{dv}{dt} = \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{dw}{dt} = \frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

or, using the laplacian notation:

$$\frac{d\mathbf{V}}{dt} = \frac{\mu}{\rho}\nabla^2 V$$

$$= \nu \nabla^2 V$$

Eddy viscosities

In analogy with the
turbulent diffusion coefficients,

We can define eddy viscosity coefficients
(or *Austauch coefficients*), $A_z \ldots$

We assume, usually, that
for horizontal processes,
A is constant, or rather, that

$$A_x = A_y = A_h$$

But that $A_h >> A_z$ because of stability.

The frictional terms are:

$$\text{friction }(x) = A_h\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + A_z\frac{\partial^2 u}{\partial z^2}$$

$$\text{friction }(y) = A_h\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + A_z\frac{\partial^2 v}{\partial z^2}$$

$$\text{friction }(z) = A_h\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + A_z\frac{\partial^2 w}{\partial z^2}$$

where

$A_h \sim 10^2 \Rightarrow 10^5\ \text{m}^2\ \text{s}^{-1}$

$A_z \sim 10^{-4} \Rightarrow 10^{-2}\ \text{m}^2\ \text{s}^{-1}$

The frictional force of the wind
on the sea surface may be written:

$$\frac{\partial \tau_x}{\partial z} = A_z \frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial \tau_y}{\partial z} = A_z \frac{\partial^2 v}{\partial z^2}$$

Reynolds stresses

See Knauss, p. 102, Box 5.5

In order to incorporate turbulent processes,
we sometimes use the "Reynolds stress" concept,
looking at velocity as the sum
of a mean flow and a turbulent flow.

Let $u = \bar{u} + u'$, where

$$\bar{u} = \frac{1}{T}\int_0^T u\,dt$$

is the *mean flow*

and

u' is the difference
between the instantaneous flow, u,
and the mean flow, $\bar{u}$
and might be regarded
as the *turbulent flow*.

By definition, the average of the
turbulent flow must be zero:

$$\bar{u}' = \frac{1}{T}\int_0^T u'\,dt = 0$$

Then, for example, the product of two
velocity components may be written:

$$uv = (\bar{u} + u')(\bar{v} + v')$$

$$= \bar{u}\,\bar{v} + \bar{u}v' + u'\bar{v} + u'v'$$

$$\overline{uv} = \overline{\bar{u}\,\bar{v}} + \overset{=0}{\cancel{\overline{\bar{u}v'}}} + \overset{=0}{\cancel{\overline{u'\bar{v}}}} + \overline{u'v'}$$

$$= \bar{u}\,\bar{v} + \overline{u'v'}$$

since $\overline{u'}$ and $\overline{v'} = 0$

and $\bar{u}$ and $\bar{v}$ are constant
over the interval T

Using this procedure, we can add terms representing the fluctuating velocity field to the equation of motion.

First, note that, for the equation of continuity,

$$\frac{\partial(\bar{u}+u')}{\partial x}+\frac{\partial(\bar{v}+v')}{\partial y}+\frac{\partial(\bar{w}+w')}{\partial z}=0$$

$$\underbrace{\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial y}+\frac{\partial\bar{w}}{\partial z}}_{\text{mean terms}}=0$$

$$\underbrace{\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}}_{\text{fluctuating terms}}=0$$

That is, both the mean and the fluctuating terms have the same form of the continuity equation

The fluctuating term (which equals zero) can be multiplied by u' and then added to the field acceleration terms in the equation of motion:

$$\underbrace{u'\frac{\partial u'}{\partial x}+v'\frac{\partial u'}{\partial y}+w'\frac{\partial u'}{\partial z}}_{\text{field acceleration terms}}$$

$$+\underbrace{u'\frac{\partial u'}{\partial x}+u'\frac{\partial v'}{\partial y}+u'\frac{\partial w'}{\partial z}}_{\text{fluctuating terms}\times u'}$$

$$=\frac{\partial}{\partial x}u'u'+\frac{\partial}{\partial y}u'v'+\frac{\partial}{\partial z}u'w'$$

Putting this term into the equation of motion and taking time averages of this expression gives:

$$\frac{d\,\overline{u}}{d\,t} = -\frac{1}{\rho}\frac{\partial\,\overline{p}}{\partial\,x} + 2\Omega\sin\phi\;\overline{v} - 2\Omega\cos\phi\;\overline{w} + \underbrace{\nu\left(\frac{\partial^2\overline{u}}{\partial\,x^2}+\frac{\partial^2\overline{u}}{\partial\,y^2}+\frac{\partial^2\overline{u}}{\partial\,z^2}\right)}_{\text{molecular frictional stresses}}$$

$$\underbrace{-\frac{\partial}{\partial\,x}\left(\overline{u'u'}\right)-\frac{\partial}{\partial\,y}\left(\overline{u'v'}\right)-\frac{\partial}{\partial\,z}\left(\overline{w'u'}\right)}_{\text{Reynolds stresses}}$$

The same thing can be done with the y- and z-components

By analogy with the molecular case,
we take the Reynolds stresses
to be related to the mean velocity
gradient by an *eddy viscosity*.

Recall the frictional terms we had earlier:

$$\text{friction }(x) = A_h\left(\frac{\partial^2 u}{\partial\,x^2}+\frac{\partial^2 u}{\partial\,y^2}\right)+A_z\frac{\partial^2 u}{\partial\,z^2}$$

$$\text{friction }(y) = A_h\left(\frac{\partial^2 v}{\partial\,x^2}+\frac{\partial^2 v}{\partial\,y^2}\right)+A_z\frac{\partial^2 v}{\partial\,z^2}$$

$$\text{friction }(z) = A_h\left(\frac{\partial^2 w}{\partial\,x^2}+\frac{\partial^2 w}{\partial\,y^2}\right)+A_z\frac{\partial^2 w}{\partial\,z^2}$$

The Reynolds stresses can be linked
to these turbulent frictional stresses,
setting, for example

$$\overline{u'u'} = -A_x\frac{\partial\,\overline{u}}{\partial\,x}$$

$$\overline{u'v'} = -A_y\frac{\partial\,\overline{u}}{\partial\,y}$$

$$\overline{u'w'} = -A_z\frac{\partial\,\overline{u}}{\partial\,z}$$

Thus, the final three Reynolds stress terms
in the equation above, become,
in the x-direction:

$$A_x\frac{\partial^2\overline{u}}{\partial\,x^2}+A_y\frac{\partial^2\overline{u}}{\partial\,y^2}+A_z\frac{\partial^2\overline{u}}{\partial\,z^2}$$

Summary

Finally, we can write,

(lumping n and A_x and calling it A_x):

$$\frac{du}{dt} = \frac{\P u}{\P t} + u\frac{\P u}{\P x} + v\frac{\P u}{\P y} + w\frac{\P u}{\P z}$$

$$= -\frac{1}{r}\frac{\P p}{\P x} + f\,v - 2\Omega\cos f\,w + A_x\frac{\P^2 u}{\P x^2} + A_y\frac{\P^2 u}{\P y^2} + A_z\frac{\P^2 u}{\P z^2}$$

$$\frac{dv}{dt} = \frac{\P v}{\P t} + u\frac{\P v}{\P x} + v\frac{\P v}{\P y} + w\frac{\P v}{\P z}$$

$$= -\frac{1}{r}\frac{\P p}{\P y} - fu + A_x\frac{\P^2 v}{\P x^2} + A_y\frac{\P^2 v}{\P y^2} + A_z\frac{\P^2 v}{\P z^2}$$

$$\frac{dw}{dt} = \frac{\P w}{\P t} + u\frac{\P w}{\P x} + v\frac{\P w}{\P y} + w\frac{\P w}{\P z}$$

$$= -\frac{1}{r}\frac{\P p}{\P z} + 2\Omega\cos f\,u - g + A_x\frac{\P^2 w}{\P x^2} + A_y\frac{\P^2 w}{\P y^2} + A_z\frac{\P^2 w}{\P z^2}$$

This is an expanded form of equation 5.28 in Knauss.

References

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