MAST602

Lecture 5

Balance of forces, effect of Earth's rotation

Hydrostatic equation
Pressure gradients in layered fluids
Terminal velocity
Coriolis force: geostrophic flow
Inertial motion
Thermal wind equation
Ekman pumping & transport
Westward intensification of ocean currents

MAST 602

Lecture 5
Balance of forces, the effect of Earth's rotation

The equations of motion,
in cartesian coordinates,
may be written:

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv - 2\Omega\cos\phi\, w + A_x\frac{\partial^2 u}{\partial x^2} + A_y\frac{\partial^2 u}{\partial y^2} + A_z\frac{\partial^2 u}{\partial z^2}$$

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu + A_x\frac{\partial^2 v}{\partial x^2} + A_y\frac{\partial^2 v}{\partial y^2} + A_z\frac{\partial^2 v}{\partial z^2}$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial z} + 2\Omega\cos\phi\, u - g + A_x\frac{\partial^2 w}{\partial x^2} + A_y\frac{\partial^2 w}{\partial y^2} + A_z\frac{\partial^2 w}{\partial z^2}$$

We will take these equations
as a starting point and examine
how they can be simplified to
represent various types of oceanic processes

In doing this, for each process we will:
- simplify
- approximate
- ignore terms

Hydrostatic equation

(Ref.: Knauss Chapter 2)

The pressure at depth z,
if density is constant, is

$$p = -\int_0^{-z} r\ g\, dz = -r\ g\ z$$

If we ignore:

- friction
- coriolis force
- acceleration,

the z-component becomes

$$\frac{1}{r}\frac{\P\, p}{\P z} = -g$$

Note, however, that in wave motion,

$$\frac{dw}{dz} \neq 0$$

so that acceleration cannot be ignored

Now, $r \approx 1027$ kg m^{-3}
and $g \approx 9.8$ m s^{-2}

Since $p = -rgz$, then we note that

10 m of seawater = 10,065 Pa
$\approx$ 1 bar

which is nearly equal to one atmosphere, i.e.,
$\sim 10^5$ Pa = 1 bar.

where 1 Pa = 1 N m^{-2}

In other words,
pressure increases about 1 atmosphere
for every 10 m of depth in the sea.

Conveniently, 1 m depth $\approx$ 1 *decibar* (dbar)

Oceanographers often express depth in dbars.

The use of decibars tends to be inaccurate
at great depths, however.

[Don't confuse decibars, dbar
with decibels, dB!]

Dynamic height (geopotential)

The work done in raising a mass M
through a vertical distance z
against the force of gravity is Mgz

This is the same as the
gain in potential energy

The geopotential, Φ, (potential
energy change per unit mass)
is defined by

$$d\Phi = g dz \ = - a dp$$

Geopotential has units of
joules kg^{-1} = m^{2} s^{-2}

Since $p = - rgz$,
the geopotential can be written as

$$d\Phi = -a \, dp$$

where $a = \frac{1}{r}$ is the specific volume.

If we integrate between two levels in the ocean:

$$\Phi_2 - \Phi_1 = \int_1^2 d\Phi$$
$$= \int_1^2 g dz$$
$$= -\int_1^2 a \, dp$$

The quantity $\Phi_2 - \Phi_1$
is called the geopotential distance
between the levels z_2 and z_1
with pressures p_2 and p_1

Geopotential "distance" has units of
energy per unit mass, J kg^{-1}

The specific volume, α, may be
expressed as

$$\alpha = \alpha_{35,0,p} + d$$

so that

$$\Phi_2 - \Phi_1 = -\int_1^2 a_{35,0,p}\, dp \ - \int_1^2 d\ dp$$
$$= -\Delta\Phi_{std} - \Delta\Phi$$

The term DF_{std} is a function of pressure only

It's called the *standard geopotential distance*

[Though it's called a *distance*,
it's really energy per unit mass]

The term $-DF$ is a function of
salinity, temperature and pressure

It's called the *geopotential anomaly*

Between two adjacent stations, a and b,
the difference in geopotential
(or *dynamic height anomaly*) is:

$$\Phi_a - \Phi_b = \int (d_a - d_b)\, dp$$

Sometimes the term *dynamic meter*,
is used to describe geopotential
(dynamic height),
1 dyn m = 10.0 J kg^{-1}

When geopotential distance is used,
the symbol D is used instead of F :

$$dD \equiv a\, dp$$

and $D = \int_{p_1}^{p_2} a\, dp$

For example,
at a depth of 100 m
i.e., z = -100 m
p = 1005 KPa = 100.5 dbar

and the geopotential distance
relative to the surface,
is $\Phi_2 - \Phi_1$= -981 J kg^{-1}

and D_2 - D_1 = – 98 dyn m.

Dynamic height is calculated
from observations data and
is related to pressure:

$$\frac{1}{r}\frac{\partial p}{\partial x} = \frac{\partial(\Delta D)}{\partial x}$$

For instance, the difference in dynamic height between adjacent stations *a* and *b* is equivalent to the difference in the horizontal pressure gradient over the same depth interval:

$$D_a = \int_{p_1}^{p_2} a_a dp$$

$$D_b = \int_{p_1}^{p_2} a_b dp$$

$$D_a - D_b = \int_{p_1}^{p_2} (a_a - a_b) dp$$

To save work when calculating by hand, a standard ocean was subtracted out, to produce the *dynamic height anomaly*:

$$\Delta D_a - \Delta D_b = \int_{p_1}^{p_2} (d_a - d_b) dp$$

With computers, this procedure is no longer necessary, but the practice persists

The politically correct terms (mostly ignored) for these quantities are:

dynamic height $\Rightarrow$ *geopotential*
dynamic height anomaly$\Rightarrow$ *geopotential difference*

Pressure gradients in layered fluids

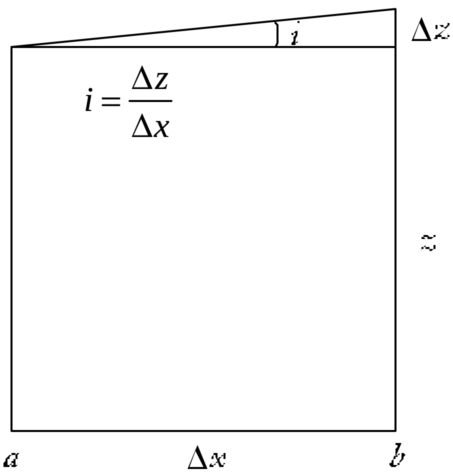


Fig 5- 1 Pressure gradient force sketch

Recall our discussion of the pressure gradient force:

$$\frac{1}{r}\frac{\P p}{\P x} = \frac{1}{r}\frac{p_b - p_a}{\Delta x}$$

$$= \frac{1}{r\Delta x}\left(r g\left(z + \Delta z\right) - r gz\right)$$

$$= g\frac{\Delta z}{\Delta x} = gi$$

Consider a two-layer fluid, where $r_2 > r_1$

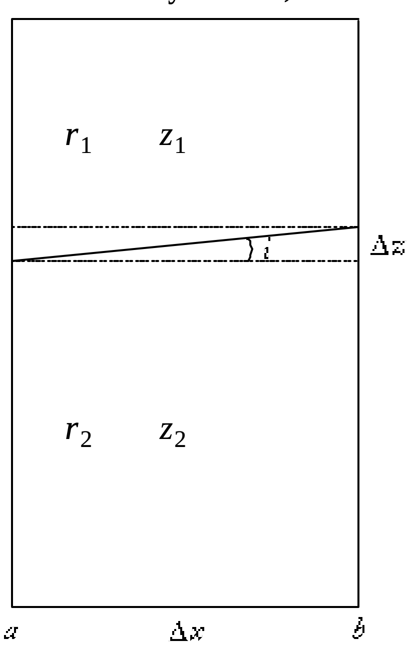


Fig 5- 2 Two-layer pressure gradient force sketch

In the lower layer,

$$\frac{1}{r}\frac{\P p}{\P x} = \frac{1}{r_2}\frac{p_b - p_a}{\Delta x}$$

$$= \frac{1}{r_2 \Delta x}\left[(r_1 g z_1 + r_2 g \Delta z + r_2 g z_2)\right.$$

$$\left. - (r_1 g z_1 + r_1 g \Delta z + r_2 g z_2)\right]$$

$$= g\frac{r_2 - r_1}{r_2}\frac{\Delta z}{\Delta x}$$

$$= g\left(\frac{r_2 - r_1}{r_2}\right) i_2$$

where i_2 = slope of the interface

The term $g\frac{r_2 - r_1}{r_2}$
is often called *reduced gravity*

If the free surface is level, there is
no horizontal pressure gradient
in the upper layer.

If the free surface slopes, then
the pressure in the lower layer is

$$\frac{1}{r_2}\frac{\P p_2}{\P x} = g\left(\frac{r_2 - r_1}{r_2}\right) i_2 + \frac{1}{r_2}\frac{\P p_1}{\P x}$$

(Yes, the subscript of r_2 in the denominator
of the second term *is* correct.)

By extension, for n layers, we can write

$$\frac{1}{r_n}\frac{\P p_n}{\P x} = g\left(\frac{r_n - r_{n-1}}{r_n}\right) i_n + \frac{1}{r_n}\frac{\P p_{n-1}}{\P x}$$

River flow

This oversimplified example shows a balance between pressure gradient force and friction

Suppose a river runs east–west (i.e., ignore the y–component)

Other assumptions are:

- ignore wind stress
- ignore coriolis force
- let acceleration = 0

Then, letting the friction be simply proportional to velocity,

$$\frac{1}{r}\frac{¶ p}{¶ x} = - J u$$

$$\frac{1}{r}\frac{¶ p}{¶ z} = - g$$

assume uniform density,

$$g\, i_x = - J u$$

e.g., consider the Mississippi River, where

$$i_x \sim 1/10{,}000 = 10^{-4}$$

and take $J \sim 10^{-3}\ \mathrm{s}^{-1}$

then

$$u = -\frac{g}{J} i_x$$

$$\approx -\frac{10\ \mathrm{ms}^{-2}}{10^{-3}\ \mathrm{s}^{-1}} \times 10^{-4}$$

$$= -1\ \mathrm{ms}^{-1}$$

That is, the speed ~ 1 m s^{-1}, or ~ 2 knots [1 knot = .5148 m s^{-1}]

Increasing the slope will increase the flow, assuming that the friction term, J, stays the same. (This is not necessarily a good assumption.)

Terminal velocity

Ref: Knauss, Chapter 6.

Let's look at a particle falling in the ocean

For this example, we assume:
- vertical motion only
- coriolis force can be ignored
- pressure gradient force can be ignored

Thus,
the balance of forces is gravitation vs. friction.

here friction ~ molecular viscosity
or ~ turbulent viscosity

the choice depends on the
size of the falling particle

The gravitational force is:

$$m\, g = (\, r_s - r_w\,)\, V\, g$$

where
r_s = density of object
r_w = density of water
V = volume of object

For a sphere,

$$mg = 4/3\, \pi\, r^3\, (r_s - r_w)\, g$$

where r = radius of the sphere

For molecular friction,
on a spherical particle,

$$F = 6\, \pi\, r\, \mu\, w$$

where μ = molecular viscosity

so,

$$w = \frac{2}{9} r^2 \frac{g}{m} (r_s - r_w)$$

This holds for Reynolds number (R_e)
less than 1, where

$$R_e = \frac{2wr\, r_w}{m}$$

i.e., this terminal velocity is valid
for particles of radius less than
a few tens of microns.

If radius > 10^{-4} m, the frictional force is

$$F = ½\, C_D\ r_w\, B\, w^2$$

where

B = cross-sectional area

C_D = drag factor
(a multidimensional factor)

C_D	object
0.1	streamlined
0.5	sphere
2	hollow hemisphere (parachute)

Then,

$$w^2 = \frac{16}{3}\,\frac{r_s - r_w}{r_w}\, gr$$

Does the small stuff ever reach the bottom?

e.g., a clay particle of 10^{-6} m radius
and specific gravity of 2.5

[specific gravity = r_S / r_W]

will have terminal velocity,

$$w = 3 \times 10^{-6}\ \mathrm{m\ s^{-1}}$$

and will take ~40 years to reach 4000 m.

How does such material get to
the ocean bottom?

= aggregation into fæcal matter.

Geostrophic Flow

The balance of forces here is
coriolis force vs. pressure gradient force.

The assumptions are:

- The flow is steady (unaccelerated)
- The flow is horizontal
- The flow is frictionless

Then the equations of motion become:

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = fv$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y} = -fu$$

Where the coriolis parameter,

$$f = 2\Omega \sin\phi$$

or, we can express the
pressure gradient forces in terms
of the sea-surface slopes,

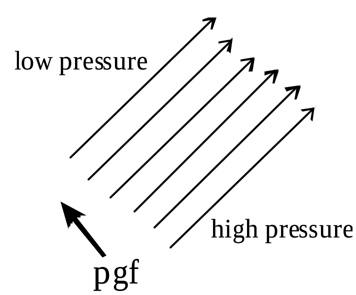
$$fv = gi_x$$

$$fu = -gi_y$$

Ocean currents that obey this relation
are called *geostrophic* currents.

The equation is extensively used
sine most major ocean currents
are geostrophic.

Note that the flow is at right angles
to the pressure gradient force.
i.e., the flow is parallel to the slope.



The forces associated with
geostrophic flow are relatively small:

Ocean currents can be maintained
by a sea-surface slope ~ $i \leq 10^{-5}$.

At sea, we can't directly measure
a slope as small as 10^{-5}.

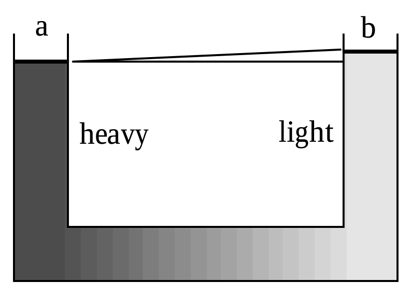
(Though satellite altimetry can
detect the gulf stream slope.)

e.g., the Gulf Stream has a height
differential of about 1 m across it
(the Sargasso Sea side is higher.)

In the atmosphere, geostrophic flow
is parallel to isobars (lines of
constant pressure).

In order to estimate currents,
we sometimes assume that
the pressure gradient force
is zero at some depth in the ocean.

In this figure,
bottom pressure at *a*
= bottom pressure at *b*



But because lighter fluid stands higher,
there will be a pressure difference
between *a* & *b* *above* the bottom.

The apparent paradox is that
the pressure is greater
where the fluid is lighter!

That is,

$$P_a\,(\text{bottom}) = P_b\,(\text{bottom})$$

but,

$$P_a\,(\text{mid depth}) < P_b\,(\text{mid depth})$$

An example of this is the Gulf Stream:
Warmer (and thus lighter) water
is on the Sargasso Sea side.

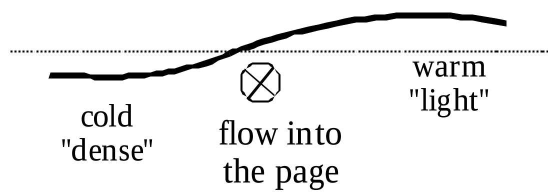
Cooler (and thus denser) water
is on the Slope Water side

Thus, the Sargasso Sea side is higher
than the Slope Water side
(by about a meter).

Furthermore, at a level below
the surface, the pressure is
greater on the Sargasso Sea side.

The pressure gradient force acts
towards the continent.

The resulting geostrophic flow,
to the right of the force,
is towards the northeast.



⊗ = flow into the page

⊙ = flow out of the page

Examples of this process can be
seen in some oceanographic
cross-sections

Fig 5- 3 Gulf Stream density section **Knauss, Fig. 6.4**

Notice the rise in sea-surface elevation
across the Gulf Stream. How can this be
measured in reality?

Another example, is the
Pacific South Equatorial Current

A complex series of zonal currents
occurs in the equatorial Pacific Ocean:

Fig 5- 4 Pacific equatorial currents, and thermocline slopes **Knauss, Fig. 6.5**

Note that, at the equator,
f changes sign
and the slope of the sea surface
associated with the geostrophic currents
also changes sign.

Barotropic and baroclinic fluids

A *barotropic fluid* is one in which the lines of constant density (*isopycnals*) do not cross lines of constant pressure (*isobars*).

A slope of the free surface, results in a horizontal pressure gradient, that is constant throughout the fluid.

In such a case, the geostrophic velocity is constant with depth.

More generally, in a multi-layer stratified fluid, if the geostrophic velocities are the same in each layer, the interfaces of all layers are parallel.

Here, isobars are parallel to isopycnals.

The multi-layer case can be generalized to the case of continuously increasing density with depth.

In a barotropic fluid,

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$$

In such a case, the vertically integrated form of the equation of continuity is:

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = 0$$

See Knauss, p. 119, Box 6.1

Divide the ocean into a mean depth, h_0 and a perturbation about the mean, h, where h is much less than h_0.

Then the integrated equation of continuity becomes

$$\frac{\partial h}{\partial t} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) h = 0$$

A baroclinic fluid is one in which the isobars and isopycnals are not parallel to one another.

This means that the geostrophic velocity changes with depth.

Looking at the Gulf Stream and Equatorial Pacific cross-sections, it's apparent that the ocean is generally highly baroclinic.

Knowing the density distribution with depth, one can calculate the *baroclinic component of the geostrophic velocity* at all depths.

To calculate the *absolute geostrophic velocity*, [i.e., barotropic plus baroclinic velocity] one needs to know the slope of the sea surface or have some independent measure of the horizontal pressure gradient of the absolute geostrophic velocity at some depth.

Inertial motion

Ref: Knauss, Chap 6

The balance of forces here is coriolis force vs. acceleration.

We make the following assumptions:
- The flow is frictionless.
- There is no pressure gradient.
- The water is somehow set in motion (by a wind, perhaps?).
- Motion is horizontal.

Then,

$$\frac{du}{dt} = fv$$

$$\frac{dv}{dt} = -fu$$

This is the equation of a circle.

Particles of fluid move in a circular path
of radius r,
at speed V,
and with a period T,

where:

$$V = \left(u^2 + v^2\right)^{\frac{1}{2}}$$

$$r = \frac{V}{f}$$

$$T = \frac{12 \text{ hours}}{\sin f}$$

$$= \frac{2p}{f}$$

T = 24 hours at 30° latitude = 12 hours at the poles = ∞ at the equator

Such motions are commonly observed
in ocean, and even lake surface waters.

Inertial motions seem to be generated
by the passage of storms and
tend to die out with time

An example of inertial oscillation
in the central Sargasso Sea:

Fig 5- 5 Inertial oscillations in a progressive vector diagram (Day and Webster 1965)

Rossby number

Whether or not the coriolis force is important can be judged by the *Rossby number*

It's the ratio of the centrifugal force to the coriolis force

The centrifugal force is related to the curvature of the flow: v^2/r

The coriolis force is vf

Thus, the ratio is:

$$R_0 = \frac{v^2/r}{vf} = \frac{v}{fr}$$

For inertial motion, the Rossby number is 1

More generally, the Rossby Number (nondimensional) is:

$$R_0 = \frac{v}{fl}$$

where l is the length scale associated with the radius of curvature

For small Rossby number, the coriolis force should be considered and the acceleration terms may often be neglected

Most oceanic flows have low Rossby number

Thermal wind equation

This topic is not treated in Knauss. For a reference, try (Pond and Pickard 1983), Sec. 8.45, *The "thermal wind" equations"*

The thermal wind equation expresses the vertical variation of the velocity field in terms of the horizontal distribution of density (or, in the atmosphere, of temperature, hence the term *thermal*).

[Obviously, from its name, this equation was developed by meteorologists.]

The assumptions are:

- No friction
- No acceleration
- Incompressible fluid

We can differentiate the geostrophic equation vertically, to get:

$$\frac{\partial(\rho f v)}{\partial z} = \frac{\partial}{\partial z}\frac{\partial p}{\partial x}$$

$$\frac{\partial\left(\rho f u\right)}{\partial z} = -\frac{\partial}{\partial z}\frac{\partial p}{\partial y}$$

If we use the hydrostatic equation,

$$\frac{\partial p}{\partial z} = -\rho g$$

then the thermal wind equations are:

$$\frac{\partial(\rho f v)}{\partial z} = -g\,\frac{\partial \rho}{\partial x}$$

$$\frac{\partial(\rho f u)}{\partial z} = g\,\frac{\partial \rho}{\partial y}$$

Note that:

- The *vertical* variation of the velocity, $\frac{\partial u}{\partial z}, \frac{\partial v}{\partial z}$ can be obtained from the *horizontal* gradients of the density field, $\frac{\partial \rho}{\partial x}, \frac{\partial \rho}{\partial y}$
- The vertical profile of *absolute* velocity can only be determined if the velocity is known at some depth

- If the lines of constant density in a fluid, (*isopycnals*) don't cross the lines of constant pressure, the *isobars*, the fluid is said to be *barotropic*. In such a case, the geostrophic velocity is constant with depth

  i.e., $\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$

- If isobars and isopycnals aren't parallel, the fluid is said to be *baroclinic* and the geostrophic velocity changes with depth

- These equations show that in the northern hemisphere, "light water is on the right" when looking in the direction of flow.

To determine the absolute value of V, there are several alternatives:

1. Assume there is a level or *depth of no motion*.
   This "reference level" is generally taken to be in deep water.
   Then velocities above that "level" can be determined.

2. If stations are available across a strait or ocean, calculate the velocities, then use the equation of continuity to see if the flow is reasonable, and adjust accordingly.

3. If current meters or floats are available, use their results as a reference velocity.

4. If altimeter measurements of the sea surface topography are available, use them to calculate the surface currents.

   (The topex/poseidon satellite, carrying a radar altimeter, was launched five years ago to support woce. It measures sea-surface topography to an accuracy of about 2 cm.)

Wind stress on the sea surface

Wind over the sea surface
induces currents in the ocean

How much?
…here are some rules of thumb:

- surface current ~ 3% of the wind
- wind stress $\propto$ square of the wind speed

Or, more formally, for the wind stress, t

$$\vec{t} = c_D r_a |\vec{u}_w| \vec{u}_w$$

or, considering only the magnitude,

$$= c_D\ r\ u_w^2$$

since
c_D = 0.95 x 10^{-3}
(the drag coefficient)

and the density of air,

r = 1.225 Kg m^{-3}

The equation can be written approximately as

$$t \cong 2\times10^{-3}\, W^2$$

with wind stress, t, [N m^{-2}]
and wind, W, [m s^{-1}]

This is a crude relationship. It varies, depending on:

- wind speed
- surface roughness
- height of W above the sea surface

W is nominally measured at a height
of 10 m, supposedly the height of a
ship's bridge above the surface of the sea

Ekman transport

Assumptions are:
- No acceleration
- No pressure gradient force
- No internal friction
- Wind stress is the only force acting

Then,

$$\frac{1}{r}\frac{¶\, t_x}{¶\, z} = -fv$$

$$\frac{1}{r}\frac{¶\, t_y}{¶\, z} = +fu$$

Now, we integrate this pair of equations
vertically to depth z,
beyond which we suppose that
the effect of the wind stress is negligible:

We get:

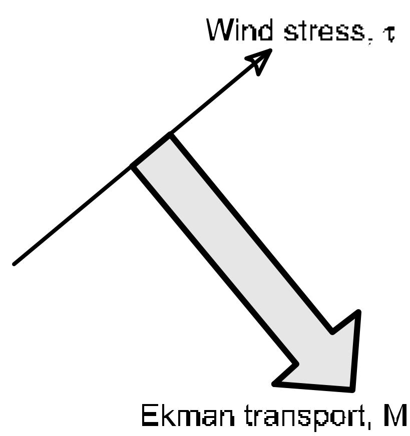
$$t_x = -M_y f$$

$$t_y = M_x f$$

where

$$M_x = \int_{-z}^{0} r\, u dz$$

$$M_y = \int_{-z}^{0} r\, v dz$$



These terms, M_x and M_y,
are transport per unit distance along the line.

z is called the *depth of the Ekman layer*

From this pair of equations we see that
wind stress moves water to the right
in the Northern Hemisphere

Ekman spiral

Ekman simplified the equation of motion
by replacing the wind stress
with an eddy viscosity.

Ekman assumed that:

- The ocean had no boundaries.
- The ocean is infinitely deep.
- Eddy viscosity coefficients are constant.
- A steady wind is blowing for a long time.
- The water is homogeneous, and the sea surface is level.
- The coriolis parameter, f, is constant.

The resulting form of the equations
is a balance between
coriolis force and friction.

The equations become, in terms of
the Ekman velocities, u_E and v_E :

$$A_z \frac{\partial^2 u_E}{\partial z^2} = f v_E$$

$$A_z \frac{\partial^2 v_E}{\partial z^2} = -f u_E$$

These two equations can be integrated
to show the details of the
vertical structure of velocity
below the surface
where the wind stress is acting:

The net transport is at right angles
to the wind vector.

The surface current is alleged
to be at 45° to the wind stress.
(It is, theoretically.)

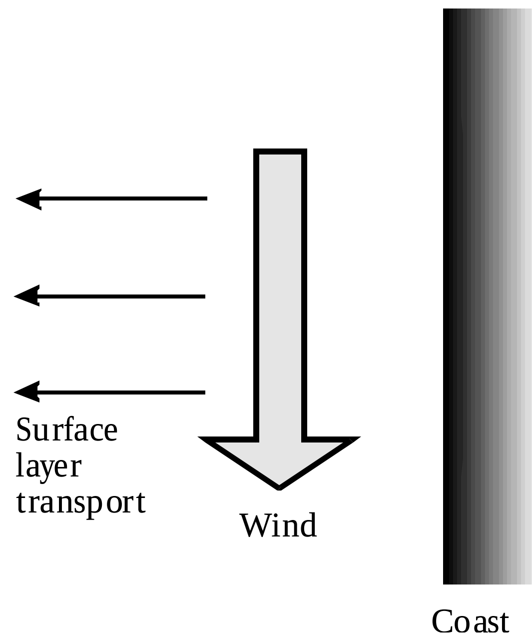
The resultant structure is known as
the *Ekman spiral.*

Is the Ekman spiral real?
= it's difficult to observe or measure.

Certainly the Ekman transport,
at right angles to the wind stress,
is a valid result.

The details of the Ekman spiral are less robust and the challenge of observing the spiral has intrigued oceanographers for decades

Upwelling (Ekman pumping)



Upwelling occurs when there is a divergent flow at the ocean surface:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}>0$$

That is, if this term is positive, then by the continuity equation, the term $\frac{\partial w}{\partial z}$ must be negative, i.e., upwelling

Conversely, if $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}$ is negative, then $\frac{\partial w}{\partial z}$ must be positive, giving downwelling

Upwelling of this type is referred to as *Ekman pumping*.

Since the upper-level ocean transport is to the right of the wind stress, this produces some interesting consequences at some coastlines and at the equator:

Coastal Upwelling

Divergent flow can occur at the surface along coastlines

Upwelling occurs along coasts:
Peru, California, Arabian Sea.

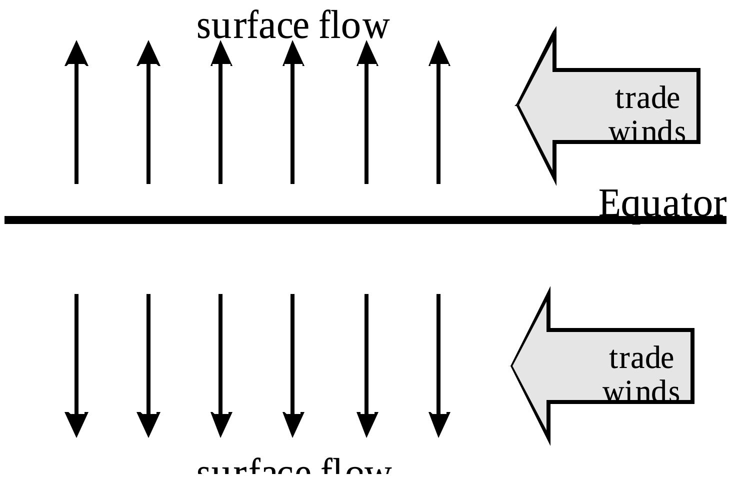
These are all regions of high biological productivity, as high-nutrient deep water comes to the surface

Equatorial Upwelling

Upwelling can also occur along the equator

Wind stress drives currents northward
in the Northern Hemisphere

And drives currents southward
in the Southern Hemisphere



The phenomenon of mid-ocean divergence
is called *Ekman pumping*.

The upwelling velocity, w_e is

$$w_e = \frac{1}{\rho f}\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right)$$

The term

$$\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}$$

is known as the *curl of the wind stress*

When it is positive, there is
divergence, and upwelling

When it is negative, there is
convergence, and downwelling

It can have a large positive value,
for example, under a hurricane

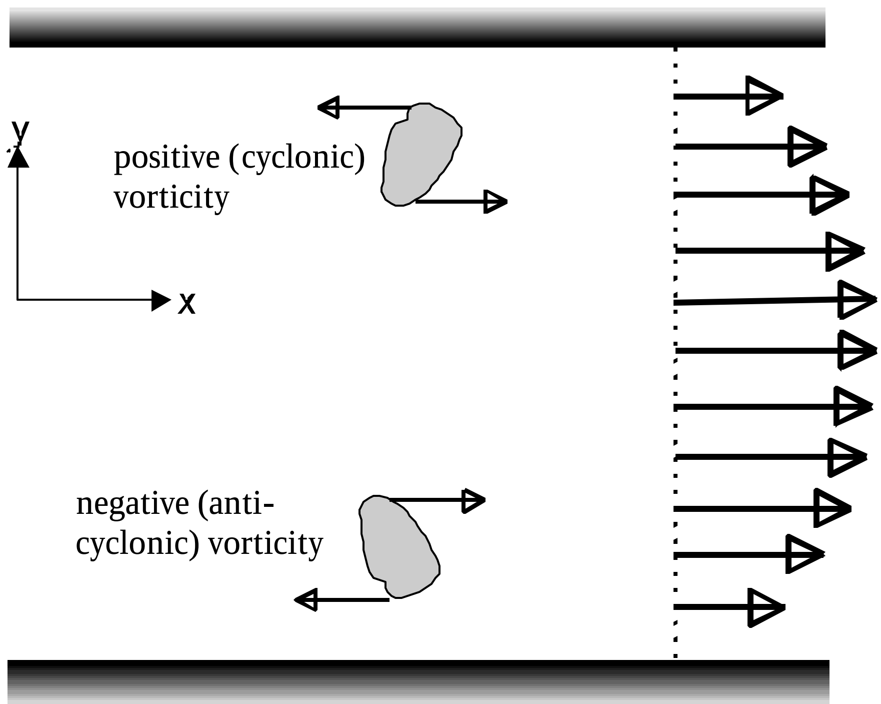
Ekman pumping occurs over the sub-tropical
gyre of the mid-North Atlantic ocean.

Vorticity

Vorticity is a measure of the spin of a particle about its axis.

It is proportional to angular momentum.
It is imparted by torques.

Example: River flow



The sign of vorticity here is

$$\frac{\partial u}{\partial y} = \text{negative} \qquad \text{(positive vorticity)}$$

$$\frac{\partial u}{\partial y} = \text{positive} \qquad \text{(negative vorticity)}$$

For this example, the vorticity in the horizontal plane is

$$x = \underbrace{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}_{\text{vorticity}}$$

Vorticity is positive when the rotation is counter-clockwise when viewed from above.

This is the direction of the rotation of the earth
as seen from above the North Pole.

positive vorticity = cyclonic rotation
negative vorticity = anti-cyclonic rotation

The term ξ is called the *relative vorticity*
because it's relative to the rotating earth.

Planetary Vorticity

Vorticity for a solid object
is $2 \times$ angular velocity.

Since the earth rotates,
at latitude f
the earth has vorticity $2\Omega \sin f$

This is called planetary vorticity. It's
the same as the coriolis parameter, f.

Thus, water that's stationary
relative to the earth
has vorticity f.

As before, f is zero at the equator.

The planetary vorticity is

$+2\Omega$ at the North Pole, and
-2Ω at the South Pole.

Absolute Vorticity

The assumptions are:

- no friction
- no vertical motion
- incompressible

The equations then are:

$$\frac{du}{dt} - fv = -\frac{1}{r}\frac{\P p}{\P x}$$

$$\frac{dv}{dt} + fu = -\frac{1}{r}\frac{\P p}{\P y}$$

We can cross-differentiate these
equations, then subtract them,
thereby eliminating pressure terms:

$$\frac{d}{d\,t}\underbrace{(x + f)}_{\substack{\text{absolute}\\ \text{vorticity}}} = -(x + f)\underbrace{\left(\frac{\P\, u}{\P\, x} + \frac{\P\, v}{\P\, y}\right)}_{\text{horizontaldivergence}}$$

$$= -(x + f)\nabla \bullet \mathbf{V}_H$$

Ref: Knauss, Box 5.6

$\mathbf{V}_H$ = horizontal velocity.

$$\nabla \bullet \mathbf{V}_H = \frac{\P u}{\P x} + \frac{\P v}{\P y}$$

= divergence of the horizontal flow.

$x + f$ = *absolute vorticity*, the sum of the relative and planetary vorticities.

This is called the *equation of conservation of absolute vorticity*.

If $\nabla \bullet \mathbf{V}_H$ is positive, i.e.,

$$\frac{\P u}{\P x} + \frac{\P v}{\P y} > 0 \text{ (divergence)}$$

the magnitude of the absolute vorticity decreases with time.

If $\nabla \bullet \mathbf{V}_H$ is negative, i.e.,

$$\frac{\P\, u}{\P\, x} + \frac{\P\, v}{\P\, y} < 0 \text{ (convergence)}$$

the magnitude of the absolute vorticity increases with time.

Usually $f > \xi$ so positive values of the absolute vorticity are usually found in the northern hemisphere, and negative values are usually found in the southern hemisphere.

Example: in a vertical cylinder of water, having planetary vorticity *f*, if fluid flow is toward the center of the cylinder, the cylinder must elongate.

Since $\nabla \bullet \mathbf{V}_H$ is negative,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} < 0,$$

the absolute vorticity, must increase.

The water will acquire some relative vorticity, x.

The absolute vorticity now is $f + x$.

Potential Vorticity

Let's consider a layer of thickness Z in the ocean.

Assume:

- Density is uniform.
- Horizontal velocity components are independent of depth

The equation of continuity may be written

$$\underbrace{\frac{1}{Z}\frac{dZ}{dt}}_{\frac{\partial w}{\partial z}} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

We can substitute $-\frac{1}{Z}\frac{\partial Z}{\partial t}$ for the horizontal divergence in

$$\frac{d}{dt}(x + f) = -(x + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

giving

$$\frac{d}{dt}(x + f) = (x + f)\left(\frac{1}{Z}\frac{dZ}{dt}\right)$$

or,

$$\frac{d}{dt}\left(\frac{x + f}{Z}\right) = 0$$

This equation says that the *potential vorticity*, $\frac{x + f}{Z}$, is constant following a particle of water.

Westward intensification of ocean currents

Ref: Knauss, p. 131

Many strong and narrow ocean currents
(like the Gulf Stream, the Kuroshio)
are found on the western side of ocean basins.

On the eastern side of ocean basins,
currents are generally broad and weak.

The pattern of large-scale horizontal circulation in the ocean basin is called a *gyre*.

It characteristically is strong and poleward in the west, and is weak and equatorward in the east.

What process can account for this western intensification?

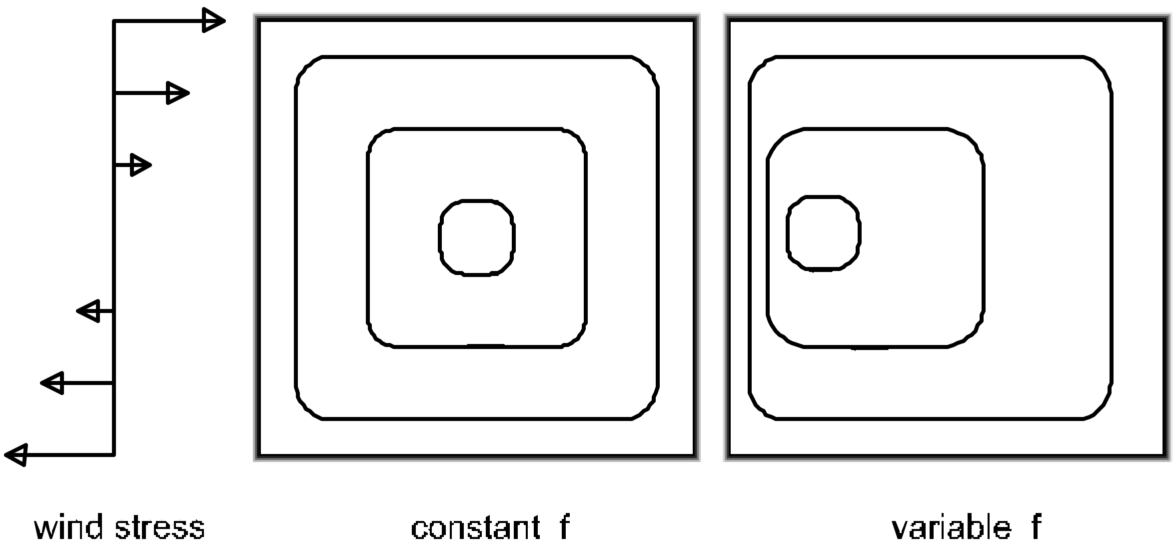
An explanation can be obtained using the conservation of vorticity.

Assume:

- Relative vorticity, x, is much less than planetary vorticity, f
- Steady state: i.e., no net change in vorticity around the gyre.
- Friction, in the form of wind stress on the sea surface must be taken into account (torques)
- Internal friction may also be possible
- Z is constant.
  (trying to include changes in Z could be complicated and doesn't change the result.)

Then $\frac{d(x + f)}{dt}$ = the sum of the frictional terms.

Consider the northern hemisphere:
- Wind is to the west in the south [trade winds]
- Wind is to the east in the north [westerlies]



Upper-layer circulation will be clockwise.

On the west side of the ocean:
- the current is northward.
- *f* will increase at higher latitude.
- relative vorticity will become more negative due to increasing *f*.
  and due to the clockwise wind stress.

On the east side of the ocean:
- the current is southward.
- *f* will decrease.
- relative vorticity will become more positive.
  due to decreasing *f*.
- relative vorticity will become more negative
  due to clockwise wind stress.

Total vorticity must balance around the circuit.
Thus we need to add vorticity in the west.

We can do this with friction: we need high friction in the west

- Suppose currents are strong and narrow in the west.

- Then, in the west, currents will
  have a strong lateral shear $\frac{\P v}{\P x}$,
  and can provide a positive vorticity
- Stronger shear means stronger
  frictional dissipation.
- Thus, there will be high friction in the west.
- In the east, currents have a weak lateral shear,
  and thus there is low friction in the east.

Thus, there is intensification of
currents in the west.

Note that the change of f with latitude
is critical with this model.

Recall that $f = 2\Omega \sin f$,
varies non-linearly with latitude.

For many problems, it's helpful
to make the variation of f linear,
thus, sometimes, we let

$$f = f_o + b\,y$$

where $b = \frac{\partial f}{\partial y}$

This approximation gives what is known
as the *beta-plane ocean*, since now the
variation of f is constant, as if it were
varying on a plane instead of a sphere.

To summarize, the balances are:

In the west:

wind stress + planetary vorticity = friction (strong currents)

In the east:

low

wind stress = planetary vorticity + friction (weak currents)

Similarly, there is westward intensification of
ocean currents in the southern hemisphere.

Lecture 5, list of figures

Fig 5- 1	Pressure gradient force sketch	
Fig 5- 2	Two-layer pressure gradient force sketch	
Fig 5- 3	Gulf Stream density section	Knauss, Fig. 6.4
Fig 5- 4	Pacific equatorial currents, and thermocline slopes	Knauss, Fig. 6.5
Fig 5- 5	Inertial oscillations in a progressive vector diagram	(Day and Webster 1965)

References

Day, C. Godfrey & Ferris Webster (1965). Some current measurements in the Sargasso Sea. *Deep-Sea Res.* **12**: 805-814.

Pond, Stephen & George L. Pickard (1983). *Introductory Dynamical Oceanography*. 2nd, Pergamon, Oxford, 329 pp.