MAST602

# Lecture 8

# Surface waves

**Wave characteristics**
**Particle motion**
**Dispersion**
**Wave energy & momentum**
**Stokes waves, capillary waves**
**Wave spectrum**
**Wave propagation**
**Refraction & breakers**
**Longshore currents & rip currenst**

# MAST 602

# Lecture 8
# Surface Waves

## Introduction

**Fig 9- 1  Wave impact at sea and on shores** **(Kinsman 1965), Photos**

> Waves exhibit both beauty and power. They can do great damage.

Surface waves have a broad spectrum:
frequencies range from $10^{-6}$ to $10^{+2}$ $s^{-1}$

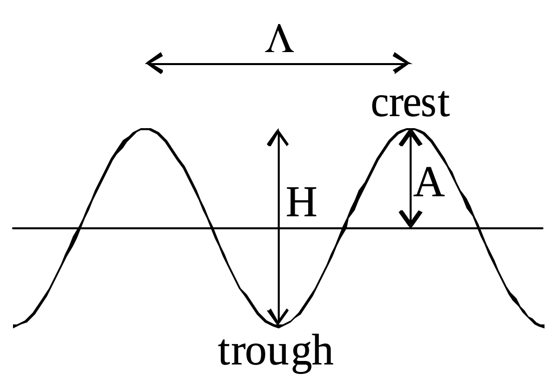
**Fig 9- 2  Wave spectrum** **(Kinsman 1965), Fig. 1.2 -1**

> Various types of ocean waves cover a range that spans many orders of magnitude. What we normally call “waves” however, are surface waves that cover a much narrower part of the spectrum.

## Wave characteristics

Here are number of terms used to describe surface wave motion:

- *period* $T$ [s]
  = time of passage of 2 successive crests
- *wavelength* $L$ [m]
  = crest-to-crest distance
- *frequency* $w = \frac{2p}{T}$ [$s^{-1}$]
- *wavenumber* $k = \frac{2p}{\Lambda}$ [$m^{-1}$
- wave *speed* $c = \frac{\Lambda}{T} = \frac{w}{k}$ [m $s^{-1}$]



- wave *height* $H$ [m]
  = trough to crest distance

- wave *amplitude* $A = \frac{H}{2}$ [m]

We distinguish between:
- the movement of the waveform
  = wave *motion*

- and the movement of the water itself
  = particle *motion*

In the following, we assume that
surface waves have a sine or cosine shape

## Small-amplitude waves

Small-amplitude is defined as
$H/\Lambda < \sim 1/20$

For a progressive wave, the vertical displacement of the free surface, $h$ in terms of the period, $T$, and wavelength, $L$, is given by:

$$h = A\cos\left[2p\left(\frac{x}{\Lambda} - \frac{t}{T}\right)\right]$$

or, using the wavenumber, $k$ and frequency, $w$, results in a nicer-looking form of the equation:

$$h = A\cos(kx - wt)$$

where
$x$ is the independent space variable
and $t$ is the independent time variable

The term $(kx - wt)$
is called the *wave phase*.

The term $(kx - wt)$ goes from 0 to $2\pi$
from one crest to the next.

The speed, $c$, is the speed at which
a point of fixed phase travels,

So $c$ is called the *phase speed*.

The phase speed, $c$,
is given by the *dispersion relation*:

$$c^2 = \frac{g}{k}\tanh kh$$

See Knauss, Box 9.1, Page 195

where $h$ is the water depth
and g is the acceleration due to gravity.

If $kh$ is small (i.e., < 0.33)
[long/shallow waves]

$\tanh kh \Rightarrow kh$

If $kh$ is large: (i.e., >3)
[short/deep waves]

$\tanh kh \Rightarrow 1$

Phase speed in these two cases becomes:

For long waves
(or shallow-water waves):

$$c_l^2 = gh \qquad c_l = \sqrt{gh}$$

Here, wave speed is controlled by depth

For short waves
(or deep-water waves):

$$c_s^2 = \frac{g}{k} \qquad c_s = \sqrt{\frac{g}{k}}$$

Here, wave speed is controlled by
wavenumber

We can write these another way
by noting that $kh = 2p\dfrac{h}{\Lambda}$

So that for long (shallow-water) waves:

$$2p\frac{h}{\Lambda} < 0.33$$

then $\frac{h}{\Lambda} < 0.05 = \frac{1}{20}$

And for short (deep-water) waves:

$$2p\frac{h}{\Lambda} > 3$$

then $\frac{h}{\Lambda} > 0.5 = \frac{1}{2}$

We simplify the dispersion relation by assuming that waves are either shallow (long) or deep (short).

> Long-wave assumption:
> wavelength at least 20 × water depth

> Short-wave assumption:
> wavelength less than 2 × water depth

Between these two limits, the dispersion relation is a lot more complex:

$$\left.\begin{aligned} c^2 &= \frac{g}{k}\tanh kh \\ w^2 &= gk\tanh kh \end{aligned}\right\} \quad \text{for } 2h < \Lambda < 20h$$

**Fig 9- 3 Wave speed vs. water depth (Pond and Pickard 1983), Fig. 12.4**

> Note that shallow-water (long) waves do not depend on wavelength; deep-water (short) waves do not depend on water depth.

## Particle motion

In deep water, particles describe circles:

the radius of the circle, $r$

is the *wave amplitude*, $A$.

The particle velocity, $u$ is the circumference of the circle, divided by the wave period, $T$,

$$u = \frac{2pr}{T}$$

**Fig 9- 4 Generation of a sinusoid (Kinsman 1965), Fig. 1.4-4**

> Circular motion (as with a particle) can generate a sinusoid (as with a surface wave form).

That is, the horizontal particle velocity, $u$, is

$$u = w\,A\cos(kx - wt)e^{kz}$$

The vertical particle velocity, $w$, is

$$w = w\,A\sin(kx - wt)e^{kz}$$

And the differential pressure due to the wave is

$$\Delta p = r\ g\,A\cos(kx - wt)e^{kz}$$

Which gives the radius

$$r = Ae^{kz}$$

And the particle velocity is

$$V = \frac{2p}{\mathrm{T}} A\,e^{kz}$$

These parameters decrease with depth:

When $z = -\frac{\Lambda}{2}$,
$r$, $V$, and $\Delta p$ are at 4%
of their surface values.

In shallow water,
particle motions are ellipses.

The *minor axis* equals
the wave amplitude at the surface
it decreases to zero at the bottom

The *major axis* doesn't vary with depth.

**Fig 9- 5 Particle paths and streamlines (Kinsman 1965), Fig. 3.3-1**

Can you convince yourself that the streamlines for the sinusoidal progressive wave on the left are consistent with the particle trajectories on the right?
Note the effects of bottom topography as the images progress from deep water to shallow water.

For a visual image of paths and streamlines,
www.coastal.udel.edu/faculty/rad/linearplot.html

**Fig 9- 6 Progressive vs. standing wave (Kinsman 1965), Fig. (Intro)**

We are all familiar with progressive waves (the ones that come ashore at the beach) but where can you find standing waves?

**Fig 9- 7 Particle paths and streamlines for a standing wave (Kinsman 1965), Fig. 3.3-2**

What are the differences between particle motions and streamlines for these standing waves compared with those we just saw for progressive waves?

**Fig 9- 8 Surface wave summary *Units & useful values* handout, p. 15**

## Wave Dispersion

Consider two sinusoidal waves:
having the same amplitude
which are superimposed

with slightly different wavenumbers
with slightly different frequencies

That is,

$$h_1 = \cos(k_1 x - w_1 t)$$

$$h_2 = \cos(k_2 x - w_2 t)$$

where $k_1 = k + \Delta k$
and $k_2 = k - \Delta k$

where $\omega_1 = \omega + \Delta\omega$
and $\omega_2 = \omega - \Delta\omega$

Then, the displacement of the free surface, $h$,
is the sum of the two components:

$$\begin{aligned} h &= h_1 + h_2 \\ &= \underbrace{2\cos(kx - wt)}_{oscillation}\underbrace{\cos(\Delta kx - \Delta wt)}_{envelope} \end{aligned}$$

with:

$$\Delta k = \frac{k_1 - k_2}{2} << k = \frac{k_1 + k_2}{2}$$

and

$$\Delta w = \frac{w_1 - w_2}{2} << w = \frac{w_1 + w_2}{2}$$

The equation for $h$, above,
describes the interaction of two waves
of nearly the same wavenumber
and of nearly the same frequency.

It gives a result where…

…the higher frequency wave, cos $(kx\text{-}\omega t)$,
with wavelength $\frac{2p}{k}$

period $\frac{2p}{w}$

phase speed $\frac{w}{k}$

…is modulated by

a lower-frequency wave, cos ($\Delta kx$ - $\Delta\omega t$)

with wavelength $\frac{2p}{\Delta k}$ ,

and period $\frac{2p}{\Delta w}$

phase speed $\frac{\Delta w}{\Delta k}$

The phase velocity of this combined wave

is known as the *group velocity*, $c_g$,

of the wave train:

$$c_g = \frac{\Delta w}{\Delta k}$$

as $k_1$ - $k_2 \Rightarrow 0$,

$$c_g = \frac{\P w}{\P k}$$

$$= \frac{d}{dk}(kc)$$

$$= \frac{c}{2}\left[1 + \frac{2kh}{\sinh kh}\right]$$

An animation of group and phase velocity can be seen at

www.phys.virginia.edu/classes/109N/more_stuff/Applets/sines/groupVelocity.html

For short (deep-water) waves,

$$c_g = \frac{1}{2}\left(\frac{g}{k}\right)^{\frac{1}{2}}$$

$$= \frac{c_s}{2}$$

For long (shallow-water) waves,

$$c_g = (gh)^{\frac{1}{2}}$$
$$= c_l$$

Wave energy travels at the group velocity

For short [deep-water] surface gravity waves, longer waves travel faster than shorter ones.
This is known as *normal dispersion*

A group of waves becomes more spread out as it travels.

Individual waves travel through the group, and disappear at the front of the group.

## Wave Energy & Momentum

The wave energy, $E$, per unit surface area in (joules m$^{-2}$)is:

$$E = \frac{1}{8} r\ g\ H^2$$
$$= \frac{1}{2} r\ g\ A^2$$

Averaged over a cycle, the wave momentum is $E/c$, so energy and momentum are linked by the phase speed

Averaged over a wavelength, wave energy of surface gravity waves is divided equally between

- potential energy
  = displacement of water from equilibrium
- kinetic energy
  = particle movement energy

## **Stokes waves** (Knauss, p. 207)

The small-amplitude gravity waves we have been discussing assume that wave height is much less than wavelength

For waves of larger height, the *Stokes wave* can be used, particularly in the case of breaking waves

Stokes waves are *trochoidal* (long troughs and sharp crests) rather than sinusoidal

The phase speed of a Stokes wave is

$$c^2 = \frac{g}{k}\left(1 + p^2 d^2\right)$$

where $d = H/L$

A typical value might be $d$ = 1/20, so the Stokes wave speed is the same as before

However, particle motion is different than with sine waves, producing a net particle speed:

$u^* = p^2 d^2 C e^{2kz}$ [Remember that $z$ is measured upward, not as in Knauss!]

e.g., for waves of period 6 and $d$ = 0.05,

$u^* \cong 0.25\, e^{2kz}$ m/s

Thus giving a significant current at the surface that drops off sharply with depth so that at 23m depth, it has dropped to $4 \times 10^{-4}$ m/s

The shallow-water form for Stokes waves is

$$C^2 \cong gh\left(1 + \frac{H}{h}\right)$$

the same as before, except in very shallow water

Waves will break when the particle speed outruns the phase speed,

$$\frac{u^*}{C} > 1$$

a condition met when the ratio of wave height to water depth exceeds ~0.7

## Wave generation and capillary waves

Wave generation mechanisms
are not well understood.

The buildup of waves in reality is more
rapid than given by theories:
e.g., a 20-knot wind gives
1 m waves in a few hours.

*Capillary waves* (surface-tension waves)
seem to play a key role
in surface-wave generation

The dispersion relation for capillary waves is

$$c^2 = \frac{g}{k} + \frac{k}{r}\Upsilon$$

where $\Upsilon$ = surface tension
= 0.05 ⇒ 0.07 N $m^{-1}$. [typical values for seawater]

Note that, for capillary waves,

- wavelengths are generally less than 5 cm
- phase speed increases with smaller wavelength
- group velocity > phase velocity
- minimum speed ~ 0.22 m $s^{-1}$
- wavelength at minimum speed ~0 0173 m (1.73 cm)

Individual capillary waves appear the
front of a group, disappear at the
back of the group.
This is known as *anomalous dispersion*.

**Fig 9- 9 Capillary-wave dispersion curve** **Knauss, Fig. 9.15**

Which part of this curve corresponds to normal dispersion? Which part corresponds to anomalous dispersion?

Slicks at sea are often areas without capillary waves
Probable cause:

- little or no wind
- oil or other material reducing surface tension
  e.g., convergence of organic material

## Wave Spectrum

Spectral analysis:
square of wave amplitude
per frequency band
= wave energy

The area under the curve
is proportional to the energy
over the frequency range

The wave spectrum is a function of:

- *wind speed* = strength of the wind
- *duration* = how long the wind's been blowing
- *fetch* = distance over which the wind is blowing

Some other wave-spectrum terms:

A *fully developed sea* is one in which
the energy imparted
to the waves by the wind
is balanced by the energy lost
through wave breaking.

The *dominant wave period*
increases as the wind increases

The *average wave height*
increases as the wind increases

Wave Statistics:

- *average wave height*
  all waves are included in the average
- *significant wave height*
  highest 1/3 of the waves
- *average of highest 10%*

## Beaufort Scale

Wave conditions can be described
in relation to a scale of winds
Sea and swell can be estimated well
by a practiced eye

**Fig 9- 10 Beaufort scale examples (Kinsman, 1965), Photo**

With some experience at sea, you can become relatively expert at judging Beaufort scale by looking at the state of the sea surface.

Examples may be found at
www.crh.noaa.gov/lot/webpage/beaufort

**Fig 9- 11 Modern Beaufort scale (von Arx 1962), Table 3-2**

In spite of modern instrumentation, this scale is still extensively used at sea. Comparison with other [electronic, say] techniques shows favorable results.

## Wave propagation

Spectral components can be added.
i.e., wave-wave interaction is generally small

Empirical relation:

$$H = H_o \cos \theta$$

where $H$ = significant wave height
$H_o$ = original significant wave height
$q$ = angle from down-range direction

At a distance
longer periods arrive first (forerunners)
energy travels at group velocity

Antarctic storms have been tracked to
waves arriving on the beach in California.

## Refraction and Breakers

The ocean bottom causes refraction
of deep-water waves as they shoal

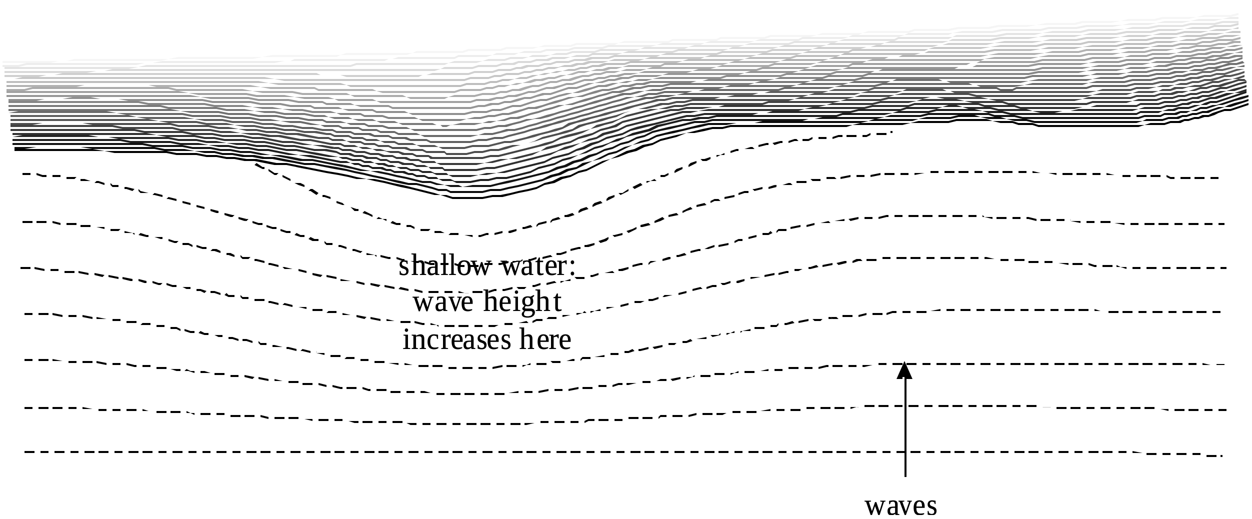
The group velocity
(and the phase velocity) is

$$\sqrt{gh}$$

Topographic refraction
is a function of wavelength
( and thus period)
i.e., longer waves feel the bottom sooner.

**Fig 9- 12 Wave refraction** **(Kinsman 1965), Fig. 1.1-3 (a, b, c)**

Underwater topography influences the refraction of waves: capes and promontories are "attacked"; coves are tranquil




As waves move onshore,
the energy flux must remain constant:

$$E = \frac{r\,g\,H^2 c_g}{8} \quad \text{joules m}^{-1}\ \text{s}^{-1}$$

$c_g$ decreases,
so $H$ must increase.

That is, wave height increases

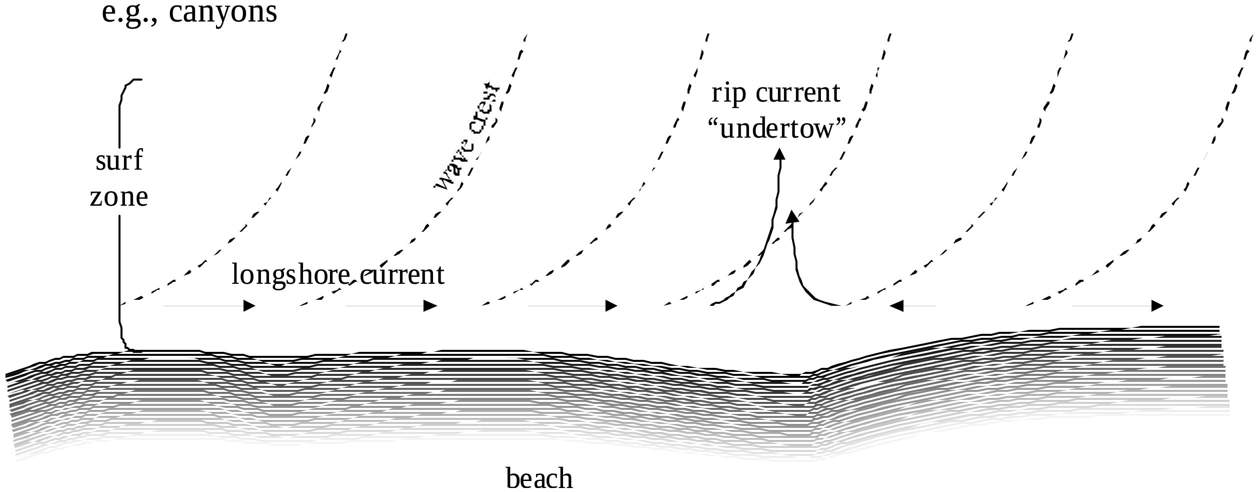
Waves break when $H \sim 0.8h$
or when $H \sim L/12$

Waves become asymmetric; break as surf

Longshore currents and rip currents
Waves breaking on a beach
transport water shoreward

If waves approach at an angle,
longshore currents are generated
in the direction of the waves

Rip currents form where
the occurrence of breaking waves
is lower than average
e.g., canyons




For parallel waves on a regular beach,
rip currents form
with spacing from 2 to 8 times
the width of the surf zone.

Rips can be ~ 3 knots (1.5 m $s^{-1}$)
these are dangerous for swimmers

If you encounter one,
don't try swimming against it!
Swim parallel to shore.

## Lecture 8 Figures



## References


Kinsman, Blair (1965). *Wind Waves*, Prentice Hall, Englewood Cliffs, NJ, 676 pp.

Pond, Stephen & George L. Pickard (1983). *Introductory Dynamical Oceanography*. 2nd, Pergamon, Oxford, 329 pp.

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